20071214 Seiberg-Witten curve 2

$$Z^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec{a}:\Lambda) = \sum_{n=0}^{\infty} \Lambda^{2nr} \,[\overline{M}(n,r)] \qquad \underset{\displaystyle\downarrow}{\mathbb{Q}(\varepsilon_1,\varepsilon_2,\vec{a})[[\Lambda]]}$$

$$Z(\varepsilon_1,\varepsilon_2,\vec{a}:\Lambda) = \text{``}\exp\Big(-\sum_{\alpha\neq\beta} \gamma_{\varepsilon_1,\varepsilon_2}(a_\alpha - a_\beta:\Lambda)\Big)\text{''}\, Z^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec{a}:\Lambda)$$

each term $\mathbb{Q}(\varepsilon_1,\varepsilon_2,\vec{a}, \log(\frac{a_\alpha - a_\beta}{\Lambda}))$

This is a formal expression.

So

$$\log Z(\varepsilon_1,\varepsilon_2,\vec{a}:\Lambda) = -\sum_{\alpha\neq\beta} \gamma_{\varepsilon_1,\varepsilon_2}(a_\alpha - a_\beta:\Lambda) + \underbrace{\log Z^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec{a}:\Lambda)}_{\mathbb{Q}(\varepsilon_1,\varepsilon_2,\vec{a})[[\Lambda]]}$$

only makes sense.

Th. $\varepsilon_1\varepsilon_2 \log Z(\varepsilon_1,\varepsilon_2,\vec{a}:\Lambda) \xrightarrow[\varepsilon_1,\varepsilon_2\to 0]{} \mathcal{F}_0$: SW prepotential

↑ holo. func. on some region.

<u>Correlation function</u>

$\mathcal{E}$: universal sheaf on $\mathbb{P}^2 \times M(n,r)$
(unique thanks to the framing)

$$\begin{array}{rccc} \mu_p : & H_*^{T^2}(\mathbb{C}^2) & \longrightarrow & H^*_{\tilde{T}}(M(n,r)) \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & \alpha & \longmapsto & \mathrm{ch}_{p+1}(\mathcal{E})/[\alpha] \end{array}$$

This gives an operator in the gauge theory.
(surface operator)

$H_*^{T^2}(\mathbb{C}^2)$ is 1-dimensional, so we just take $[\mathbb{C}^2]$.

(Since $\mathbb{C}^2$ is noncompact, $[\mathbb{C}^2]$ must be defined via the localized equiv. homology

There are three other natural classes:

$$\begin{array}{ccc} O : \text{origin}, & \{x=0\}, & \{y=0\} \\ \| & \| & \| \\ \varepsilon_1\varepsilon_2\cap[\mathbb{C}^2] & \varepsilon_2\cap[\mathbb{C}^2] & \varepsilon_1\cap[\mathbb{C}^2] \end{array}$$

$\vec{\tau}=(\tau_1,\tau_2,\tau_3,\cdots)$: formal variables

$$Z^{inst}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\tau};\Lambda) = \sum_{n=0}^{\infty}\Lambda^{2nr}\int_{M(n,r)}\exp\left(\sum_{p=1}^{\infty}\tau_p\,\mu_p([\mathbb{C}^2])\right)$$

This gives all correlation functions as

$$\frac{\partial^p}{\partial\tau_{i_1}\cdots\partial\tau_{i_p}}Z^{inst}\Big|_{\vec{\tau}=0} = \sum\Lambda^{2nr}\int_{M(n,r)}\mu_{i_1}([\mathbb{C}^2])\cdots\cdots\mu_{i_p}([\mathbb{C}^2])$$

$$=\langle\mu_{i_1}([\mathbb{C}^2])\cdots\cdots\mu_{i_p}([\mathbb{C}^2])\rangle$$

Rem. ∘ $c_1(\mathcal{E}) = 0 \Longrightarrow$ we do not need to introduce μ_0

∘ $$ch_2(\mathcal{E})/[\mathbb{C}^2] = \underset{\substack{\uparrow \\ ch_2(\text{trivial}) \curvearrowleft T^{r-1}}}{\frac{1}{\varepsilon_1\varepsilon_2}\sum_{\alpha=1}^{r} a_\alpha^2} - n$$

So $$\int_{M(n,r)} \exp\left(\tau_1 \mu_1([\mathbb{C}^2])\right) \cdots$$

$$= e^{\frac{\tau_1}{\varepsilon_1\varepsilon_2}\sum a_\alpha^2} \boxed{e^{-n\tau_1}} \int_{M(n,r)} \cdots$$

This can be absorbed into Λ

i.e. τ_1 ····· essentially $-\frac{1}{2r}\log\Lambda$

(Strategy of the proof of Th)

- How to see the SW curve?

Consider correlation functions on the blowup $\hat{\mathbb{C}}^2$.

$\rightsquigarrow$ localization formula gives us a simple formula interms of Z on $\mathbb{C}^2$

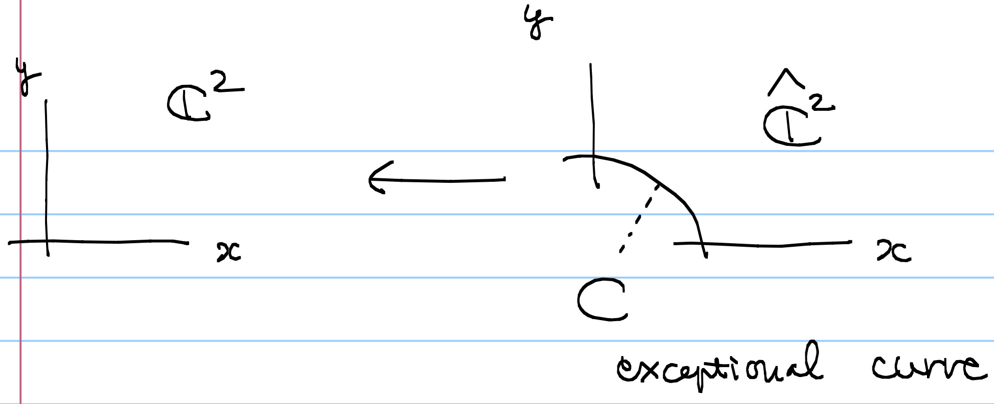
(It involves Riemann theta function for SW curve in the limit $\varepsilon_1, \varepsilon_2 \to 0$)

On the other hand, a simple dimension counting argument gives some corr. functions $= 0$

$\Longrightarrow$ Z satisfies a certain equation, which characterise it.

$\Longrightarrow$ $\lim_{\varepsilon_1, \varepsilon_2 \to 0} \varepsilon_1 \varepsilon_2 \log Z$ satisfies a diff. equ.

SW prepotential satisfies the same differential equation.



$\hat{M}(k,n,r)$ = framed moduli space of sheaves $\hat{E}$ on $\hat{\mathbb{P}}^2$

$$\langle c_1(\hat{E}), [C] \rangle = -k$$
$$\langle c_2(\hat{E}) - \frac{r-1}{2r} c_1(\hat{E})^2, [\hat{\mathbb{P}}^2] \rangle = n$$

$\hat{\mathcal{E}}$: universal sheaf, ⇝ μ as before.
Then it is natural to consider $\mu_p([C])$

$$\hat{Z}^{inst}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\tau}, \vec{t})$$

$$\overset{\text{def.}}{=} \sum_n \Lambda^{2nr} \int_{\hat{M}(k,n,r)} \exp\left[\sum t_p \mu_p([C]) + \tau_p \mu_p([\hat{\mathbb{C}}^2])\right]$$

Fixed pts :

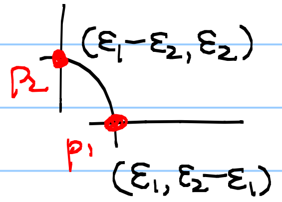
$$\hat{M}(k,n,r)^T = \{ E_1 \oplus \cdots \oplus E_r \mid \underset{\displaystyle \overset{\|}{\mathcal{O}(k_\alpha C)\otimes \underset{\vdots}{\mathcal{I}_\alpha}}}{E_\alpha} : \text{rk } 1 \}$$

The twist by a line bundle is a new feature! $\rightsquigarrow$ $\mathcal{O}(k_\alpha C)\otimes \mathcal{I}_\alpha$

ideal sheaf

$$\hat{M}(k,n,r)^{T\times\mathbb{C}^*\times\mathbb{C}^*} = \{ \bigoplus_\alpha \mathcal{O}(k_\alpha C)\otimes \mathcal{I}_\alpha \mid \mathcal{I}_\alpha \text{ is fixed by } \mathbb{C}^*\times\mathbb{C}^* \}.$$

Supp $\mathcal{O}/\mathcal{I}_\alpha$: $(\hat{\mathbb{C}}^2)^{\mathbb{C}^*\times\mathbb{C}^*}$ = 2 points p_1, p_2



$$\mathcal{I}_\alpha = \mathcal{I}_\alpha^1 \cap \mathcal{I}_\alpha^2$$

And $\mathcal{I}_\alpha^1, \mathcal{I}_\alpha^2$ are given by monomial ideals in the toric coordinate at p_1, p_2.

○ the tangent space at the fixed point :

$$\bigoplus_{\alpha,\beta} \mathrm{Ext}^1(\mathcal{I}_\alpha(k_\alpha C), \mathcal{I}_\beta(k C - \ell_\infty))$$

$$
\begin{aligned}
= \bigoplus_{\alpha\beta} & \mathrm{Ext}^1(\mathcal{O}(k_\alpha C), \mathcal{O}(k_\beta C - \ell_\infty)) \\
& \oplus \mathrm{Ext}^1(\mathcal{I}^1_\alpha(k_\alpha C), \mathcal{I}^1_\beta(k_\beta C - \ell_\infty)) \leftarrow P_1 \\
& \oplus \mathrm{Ext}^1(\mathcal{I}^2_\alpha(k_\alpha C), \mathcal{I}^2_\beta(k_\beta C - \ell_\infty)) \leftarrow P_2
\end{aligned}
$$

Therefore the equiv. Euler class is

$$
\begin{aligned}
& \prod e(\mathrm{Ext}^1(\mathcal{O}(k_\alpha C), \mathcal{O}(k_\beta C - \ell_\infty)) \\
& \times e(T_{\oplus_\alpha \mathcal{I}^1_\alpha} M(n_1, r))\Big|_{\substack{\varepsilon_1 \to \varepsilon_1 \\ \varepsilon_2 \to \varepsilon_2 - \varepsilon_1 \\ a_\alpha \to a_\alpha + \varepsilon_1 k_\alpha}} e(T_{\oplus_\alpha \mathcal{I}^2_\alpha} M(n_2, r))\Big|_{\substack{\varepsilon_1 \to \varepsilon_1 - \varepsilon_2 \\ \varepsilon_2 \to \varepsilon_2 \\ a_\alpha \to a_\alpha + \varepsilon_2 k_\alpha}}
\end{aligned}
$$

From the 2nd & 3rd term, we get the partition function Z on $\mathbb{R}^4$.

We need to compute the 1st term. Then we find that it is absorbed into the perturbation part.

We then get

$$
\boxed{
\begin{aligned}
& \hat{Z}^{c_1=k}(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{t}, \vec{\tau}; \Lambda) \\
& = \sum_{\vec{k} \in \underset{\substack{\uparrow \\ \text{need to be shifted} \\ \text{if } \vec{k} \neq 0}}{\mathbb{Z}^{r-1}}} Z(\varepsilon_1, \varepsilon_2 - \varepsilon_1, \vec{a} + \varepsilon_1 \vec{k}, \vec{\tau} + \varepsilon_1 \vec{t}; \Lambda) \\
& \qquad \times Z(\varepsilon_1 - \varepsilon_2, \varepsilon_2, \vec{a} + \varepsilon_2 \vec{k}, \vec{\tau} + \varepsilon_2 \vec{t}; \Lambda)
\end{aligned}
}
\tag{*}
$$

__Rem__ ① We have used the Chern character in the definition of μ_p.
This is because $ch(\mathcal{E}+\mathcal{F}) = ch(\mathcal{E}) + ch(\mathcal{F})$
and the above formula is simplified.

In the K-theoretic version, the natural generalization seems ψ_k : Adams operator.
But as they don't form an integral base,
It may be better to consider Schur functor.
(power sum v.s. Schur func.)

⋮ ⋮

not int. base integral base.

② exp(perturb.) was a formal expr.
But in the above formula
(pert. part in the left hand side)
− (pert. right 〃)
can be exponentiated ?

So the above formula should be understood in this way.

○ Dimension counting argument:

Prop. $\int_{\hat{M}(0,n,r)} \mu_1(C)^d = 0 \qquad d = 1, \cdots, 2r-1$

Rem. This was well-known in the ordinary Donaldson invariants.

(proof)

$$\exists\ \hat{\pi} : \hat{M}(0,n,r) \longrightarrow \overline{M}(n,r) \qquad p : \hat{\mathbb{C}}^2 \to \mathbb{C}^2$$

$$\begin{tikzcd} \hat{E} \arrow[r, mapsto] & (p_* \hat{E}^{\vee\vee} + \text{singularities}) \end{tikzcd}$$

This is an isomorphism on the open set $M^{\text{l.f.}}(n,r)$.

$\mu_1(C)$ can be represented by a divisor
s.t $\hat{\pi}(\mu_1(C)) \subset \overline{M}(n-1,r) \times \{0\}$

$$\begin{array}{c} \overline{M}(n-1,r)\times\{0\} \\ \cap \\ \overline{M}(n,r) \end{array}$$

∵ $\hat{E} \in \mu(C)$ must have singularities along C //

$$\therefore\ \hat{\pi}_*(\mu_1(C)^d \cap [\hat{M}]) \in \mathrm{Im}\big(\underbrace{H^T_*(\overline{M}(n-1,r)\times\{0\})}_{\text{codim} = 2r} \to H^T_*(\overline{M}(n,r))\big)$$

//

Consequence: Put $\vec{\tau} = \vec{0}$, $\vec{t} = (t_1, 0, 0, \cdots)$ in $(*)$

Then Coeff. of t_1^d of RHS of $(*) = 0$

$d = 1, \cdots, 2r-1$

More generally the same is true for

degree in $\vec{t}$ $= 1, \cdots, 2r-1$ $\left(\begin{array}{l} \deg t_1 = 1 \\ \quad t_2 = 2 \\ \quad \vdots \end{array} \right)$

And $d = 0 \Rightarrow$ get Z.

This determine $Z(\varepsilon_1, \varepsilon_2, \vec{a}; \Lambda)$

recursively in n $(n = 0 \cdots M(0, r) = pt)$

From this it is not difficult to prove

$\varepsilon_1\varepsilon_2 \log Z$ is regular at $\varepsilon_1 = \varepsilon_2 = 0$

$$\varepsilon_1\varepsilon_2 \log Z = F(\vec{a}, \vec{\tau}; \Lambda) + (\varepsilon_1 + \varepsilon_2) H(\vec{a}, \vec{\tau} : \Lambda)$$
$$+ \varepsilon_1\varepsilon_2 A(\vec{a}, \vec{\tau} ; \Lambda) + \frac{\varepsilon_1^2 + \varepsilon_2^2}{3} B(\vec{a}, \vec{\tau} : \Lambda) + \text{higher}$$

From the recursive equation, it is also

easy to show $H(\vec{a}, \vec{\tau} : \Lambda) = \frac{\pi\sqrt{-1}}{2} \sum_{\alpha<\beta} (a_\beta - a_\alpha)$.

We expand the RHS of $(*)$ after setting $\vec{\tau}=\vec{0}$

$$\frac{F(\vec{a}+\varepsilon_1\vec{k},\ \varepsilon_1\vec{t})}{\varepsilon_1(\varepsilon_2-\varepsilon_1)}+\frac{F(\vec{a}+\varepsilon_2\vec{k},\ \varepsilon_2\vec{t})}{(\varepsilon_1-\varepsilon_2)\varepsilon_2}$$

$$=\frac{F(\vec{a},\vec{0})}{\varepsilon_1\varepsilon_2}-\left[\frac{\partial^2 F}{\partial\tau_p\partial\tau_q}\frac{t_pt_q}{2}+\frac{\partial^2 F}{\partial\tau_p\partial a^i}t_pk^i+\frac{\partial^2 F}{\partial a^i\partial a^j}\frac{k^ik^j}{2}\right]$$

$$+\text{higher}$$

where $a^i=a_i-a_{i+1}\quad(i=1,\cdots,r-1)$

$$\frac{\varepsilon_2 H(\vec{a}+\varepsilon_1\vec{k},\varepsilon_1\vec{t})}{\varepsilon_1(\varepsilon_2-\varepsilon_1)}+\frac{\varepsilon_1 H(\vec{a}+\varepsilon_2\vec{k},\varepsilon_2\vec{t})}{(\varepsilon_1-\varepsilon_2)\varepsilon_2}$$

$$=\frac{(\varepsilon_1+\varepsilon_2)H(\vec{a},\vec{0})}{\varepsilon_1\varepsilon_2}+\left[\underset{\substack{\shortparallel\\0}}{\frac{\partial H}{\partial\tau_p}}t^p+\underbrace{\frac{\partial H}{\partial a^i}}_{\substack{\shortparallel\\-\pi i\langle\vec{k},\rho\rangle}}k^i\right]+\text{higher}\qquad\left(\rho=\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha\right)$$

$$A(\vec{a}+\varepsilon_1\vec{k},\varepsilon_1\vec{t})+A(\vec{a}+\varepsilon_2\vec{k},\varepsilon_2\vec{t})=2A(\vec{a},\vec{0})+\text{higher}$$

$$\frac{\varepsilon_1^2+(\varepsilon_2-\varepsilon_1)^2}{3}\frac{B(\vec{a}+\varepsilon_1\vec{k},\varepsilon_1\vec{t})}{\varepsilon_1(\varepsilon_2-\varepsilon_1)}+\frac{(\varepsilon_1-\varepsilon_2)^2+\varepsilon_2^2}{3}\frac{B(\vec{a}+\varepsilon_2\vec{k},\varepsilon_2\vec{t})}{(\varepsilon_1-\varepsilon_2)\varepsilon_2}$$

$$=\frac{\varepsilon_1^2+\varepsilon_2^2-3\varepsilon_1\varepsilon_2}{3\varepsilon_1\varepsilon_2}B(\vec{a},\vec{0})+\text{higher}$$

We divide RHS by Z

$$\therefore \sum_{\vec{k}} \exp\left[-\frac{\partial^2 F}{\partial \tau_p \partial \tau_q} \frac{t_p t_q}{2} - \frac{\partial^2 F}{\partial \tau_p \partial a^i} t_p k^i - \frac{\partial^2 F}{\partial a^i \partial a^j} \frac{k^i k^j}{2} \right]$$

$$(-1)^{\langle \vec{k}, p \rangle} \exp(A - B) = 1 + O(x^{2r})$$

$$\tau_{ij} = -\frac{1}{2\pi\sqrt{-1}} \frac{\partial^2 F}{\partial a^i \partial a^j}$$ (period matrix of SW curve)

$\sum_{\vec{k}}$ $\rightsquigarrow$ Riemann theta function Θ_E

characteristic E is related $(-1)^{\langle \vec{k}, p \rangle}$

- constant in t

$$\Theta_E(0;\tau) = \exp(B - A)$$

- coeff. of $t_p t_q$: $(p+q \leqq 2r-1)$

$$-\frac{\partial^2 F}{\partial \tau_p \partial \tau_q} + \frac{1}{\pi\sqrt{-1}} \sum_{i,j} \frac{\partial^2 F}{\partial \tau_p \partial a^i} \cdot \frac{\partial^2 F}{\partial \tau_q \partial a^j} \cdot \frac{\partial}{\partial \tau_{ij}} \log \Theta_E(0,\vec{\tau}) = 0$$

(contact term equation)

$p = q = 1$ SW prepotential $\mathcal{F}$ satisfies the same equation.

Also $\frac{\partial F}{\partial \tau_p}$ $(p = 1, \cdots, r-1)$ are essentially u_α : coeff. of SW curve.